MATHEMATICS

NOTE ON A PAPER BY BERNŠTEĬN, GEL'FAND AND GEL'FAND ON VERMA MODULES

BY

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Let $\mathfrak{G}$ denote a complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{G}$ with dual $\mathfrak{h}^*$. In [6] VERMA studied "universal modules with a highest weight" M_λ, where $\lambda \in \mathfrak{h}^*$. (For a precise definition of M_λ see below.) He proved that $M_{s_\alpha(\lambda)} \subset M_\lambda$ if $\lambda - s_\alpha(\lambda) = n\alpha$ for some non-negative integer n, and conjectured that every imbedding $M_\mu \subset M_\lambda$ is a composition of imbeddings of that type. This conjecture was proved by I. N. BERNŠTEĬN, I. M. GEL'FAND and S. I. GEL'FAND in [2], in a rather awkward way; their proof doesn't get any simpler if λ is assumed to be integral. In this note a straightforward fundamentally simple proof of the conjecture is given for integral λ, (this is theorem 3).

Let R denote the root system of $\mathfrak{G}$ with respect to $\mathfrak{h}$ and $\mathfrak{G}^\alpha$ the root space corresponding to a root α. Let S be a fundamental system in R. This induces an ordering in R: $R = R_+ \cup R_-$; we set $\mathfrak{N}_\pm = \sum_{\alpha \in R_\pm} \mathfrak{G}^\alpha$. Let $(,)$ be the inverse Killing form on $\mathfrak{h}^*$; then

$$\mathfrak{h}^*_{\mathbf{Z}} = \left\{ \lambda \in \mathfrak{h}^* \,\middle|\, \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbf{Z}, \text{ for all } \alpha \in R \right\}$$

is the lattice of integral elements in $\mathfrak{h}^*$. Let $C = \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha) > 0 \text{ if } \alpha \in R_+\}$ be the highest Weyl chamber with respect to S and $\bar{C}$ its closure. W denotes the Weyl group of $\mathfrak{G}$ and for $\alpha \in R$, $s_\alpha \in W$ is defined by

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha \text{ for } \lambda \in \mathfrak{h}^*.$$

For $w \in W$, $\mathfrak{l}(w)$ denotes the length of w. In [1, Chap. VI, no. 1.6, prop. 17] the following is proved:

(*) if $\alpha \in R_+$ then $\mathfrak{l}(s_\alpha w) > \mathfrak{l}(w)$ if and only if $w^{-1}(\alpha) \in R_+$.

For $\alpha \in R_+$, we define $H_\alpha \in \mathfrak{h}$ by

$$\lambda(H_\alpha) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \text{ for all } \lambda \in \mathfrak{h}^*.$$

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It is possible to choose $X_\alpha \in \mathfrak{G}^\alpha$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$ for all $\alpha \in R$.

Let $U(\mathfrak{G})$ denote the universal enveloping algebra of $\mathfrak{G}$ and $U(\mathfrak{N}_-)$ the subalgebra of $U(\mathfrak{G})$ generated by 1 and $\mathfrak{N}_-$. Let $\delta = \sum_{\alpha \in R_+} \frac{1}{2}\alpha$; then $\delta \in \mathfrak{h}^*_{\mathbf{Z}}$.

For $\lambda \in \mathfrak{h}^*$ we define I_λ as the left ideal of $U(\mathfrak{G})$ generated by $\mathfrak{N}_+$ and $\{H - (\lambda - \delta)(H) | H \in \mathfrak{h}\}$. Then $U(\mathfrak{G}) = U(\mathfrak{N}_-) \oplus I_\lambda$. The left $U(\mathfrak{G})$-module $U(\mathfrak{G})/I_\lambda$ is denoted by M_λ; M_λ has a highest weight vector $f_\lambda = 1 + I_\lambda$ with weight $\lambda - \delta$.

THEOREM 1 (Verma). Let $\lambda, \mu \in \mathfrak{h}^*$. Only two cases are possible:
(i) $\mathrm{Hom}_{U(\mathfrak{G})}(M_\mu, M_\lambda) = 0$;
(ii) $\dim_{\mathbf{C}} \mathrm{Hom}_{U(\mathfrak{G})}(M_\mu, M_\lambda) = 1$; every non-trivial homomorphism is an imbedding. Moreover, there is a $w \in W$ such that $w(\mu) = \lambda$. (For a proof, see [6].)

Let $\mu, \lambda \in \mathfrak{h}^*$ and suppose $u \in U(\mathfrak{N}_-)$, $u \neq 0$ and $I_\mu u \subset I_\lambda$, then the right multiplication with u in $U(\mathfrak{G})$, induces a non-trivial homomorphism $M_\mu \to M_\lambda$ and hence an imbedding, which is denoted by $r(u)$. If, for some $u \in U(\mathfrak{N}_-)$ uf_λ is $\mathfrak{N}_+$-extreme (i.e. annihilated by $\mathfrak{N}_+$), with weight $\mu - \delta$, then $r(u)\colon M_\mu \to M_\lambda$ is an imbedding.

LEMMA 1. For all $\alpha \in R$ and k a nonnegative integer, $[X_\alpha, X^k_{-\alpha}] = -kX^{k-1}_{-\alpha}(H_\alpha - k + 1)$. If $\alpha \in S$ and $\lambda \in \mathfrak{h}^*$ are such that $\lambda(H_\alpha)$ is a nonnegative integer, then $X^{\lambda(H_\alpha)}_{-\alpha} f_\lambda$ is $\mathfrak{N}_+$-extreme with weight $s_\alpha(\lambda) - \delta$.

PROOF. The first statement follows by induction on k. The second statement immediately follows from the first.

Let $Q_L(\mathfrak{N}_-)$ denote the left division ring of quotients of $U(\mathfrak{N}_-)$; see [3, p. 166] for its definition and existence. By $\mathbf{P}(Q_L(\mathfrak{N}_-))$ we denote the set of all one-dimensional subspaces of $Q_L(\mathfrak{N}_-)$. We define a multiplication in $\mathbf{P}(Q_L(\mathfrak{N}_-))$ as follows: $\mathbf{C}X \cdot \mathbf{C}Y = \mathbf{C}X \cdot Y$ for all $X, Y \in Q_L(\mathfrak{N}_-)$, $X \neq 0$, $Y \neq 0$.

THEOREM 2 (Verma). There exists a map $u_L\colon W \times \mathfrak{h}^*_{\mathbf{Z}} \to \mathbf{P}(Q_L(\mathfrak{N}_-))$ with:
(i) $u_L(s_\alpha, \lambda) = \mathbf{C}X^{\lambda(H_\alpha)}_{-\alpha}$, if $\alpha \in S$;
(ii) $u_L(w_1 w_2, \lambda) = u_L(w_1, w_2(\lambda)) \cdot u_L(w_2, \lambda)$.
If $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$, and $w \in W$, then $M_{w(\lambda)} \subset M_\lambda$ if and only if $u_L(w, \lambda) \subset U(\mathfrak{N}_-)$. In fact, $\mathrm{Hom}_{U(\mathfrak{G})}(M_{w(\lambda)}, M_\lambda) = r(u_L(w, \lambda))$, if $u_L(w, \lambda) \subset U(\mathfrak{N}_-)$.

PROOF. This theorem is a weak version of theorems 4 and 5 in [6], which are harder to prove but not needed for our purpose.

In the same way it can be proved that there exists a map $u_R\colon W \times \mathfrak{h}^*_{\mathbf{Z}} \to \mathbf{P}(Q_R(\mathfrak{N}_-))$, where $Q_R(\mathfrak{N}_-)$ is the right division ring of quotients, with $u_R(s_\alpha, \lambda) = \mathbf{C}X^{\lambda(H_\alpha)}_{-\alpha}$ for $\alpha \in S$, and $u_R(w_1 w_2, \lambda) = u_R(w_1, w_2(\lambda)) \cdot u_R(w_2, \lambda)$. Then one has also for $w \in W$, $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$: $M_{w(\lambda)} \subset M_\lambda$ if and only if $u_R(w, \lambda) \subset U(\mathfrak{N}_-)$.

If J is a subset of S, let W_J be the subgroup of W generated by the s_α

with $\alpha \in J$. Let $D_J = \{w \in W | \mathfrak{l}(ws_\alpha) > \mathfrak{l}(w)$, for all $\alpha \in J\}$, then D_J is a set of left coset representatives of W/W_J, see [1, exercise 3 of p. 37].

LEMMA 2 (Steinberg). Let $\lambda \in \mathfrak{h}^*$ and $S(\lambda) = \{\alpha \in S | (\lambda, \alpha) = 0\}$. Then $W_{S(\lambda)}$ is the stabilizer of λ in W. Suppose $\lambda \in \bar{C}$, $d \in D_{S(\lambda)}$ and $\alpha \in S$; if $d(\lambda)(H_\alpha) < 0$, then $\mathfrak{l}(s_\alpha d) < \mathfrak{l}(d)$ and $s_\alpha d \in D_{S(\lambda)}$.

PROOF. The first statement is proved in [4, lemma 1.18]. If $\lambda \in \bar{C}$, $d \in D_{S(\lambda)}$, $\alpha \in S$ and $d(\lambda)(H_\alpha) < 0$, it follows from (*) that $\mathfrak{l}(s_\alpha d) < \mathfrak{l}(d)$ and from the definition of $D_{S(\lambda)}$ that $s_\alpha d \in D_{S(\lambda)}$.

The Bruhat ordering in W is defined as follows: If $w_1, w_2 \in W$, then $w_1 \leqslant w_2$ if there exist $\alpha_1, \dots \alpha_k \in R$, such that $\mathfrak{l}(w_1 s_{\alpha_1} \dots s_{\alpha_{i-1}}) < \mathfrak{l}(w_1 s_{\alpha_i} \dots s_{\alpha_i})$ for $i = 1, \dots, k$, and $w_1 s_{\alpha_1} \dots s_{\alpha_k} = w_2$. See [7].

Then $w_1 \leqslant w_2$ if and only if for every reduced expression $w_2 = s_{\alpha_1} \dots s_{\alpha_l}$, $\alpha_1, \dots, \alpha_l \in S$, there exists a subsequence $(i_1, \dots, i_p)$ of $(1, \dots, l)$ such that $w_1 = s_{\alpha_{i_1}} \dots s_{\alpha_{i_p}}$. See [5, exercise on p. 128] or [7].

Then it is clear that:

(**) if $w_1, w_2 \in W$ with $w_1 \leqslant w_2$ and $\mathfrak{l}(s_\alpha w_1) > \mathfrak{l}(w_1)$, $\mathfrak{l}(s_\alpha w_2) > \mathfrak{l}(w_2)$ for some $\alpha \in S$, one has $s_\alpha w_1 \leqslant s_\alpha w_2$.

See also [7] for a more general result.

Let $w_0 \in W$ be the longest element of W. Then $S \ni \alpha \mapsto -w_0(\alpha) \in S$, defines an involutory automorphism of the Dynkin diagram, which corresponds to an involutory automorphism A of $\mathfrak{G}$, which leaves $\mathfrak{N}_-$ invariant. Then θ: $\mathfrak{N}_- \to \mathfrak{N}_-$ defined by $\theta(X) = -A(X)$ for $X \in \mathfrak{N}_-$, is an anti-automorphism. This can be uniquely extended to an anti-automorphism of $U(\mathfrak{N}_-)$, which is again denoted by θ. It then follows from the definition of left and right division ring of quotients, that:

$$Q_L(\mathfrak{N}_-) \ni \frac{a}{b} \mapsto \frac{\theta(a)}{\theta(b)}{}_i \in Q_R(\mathfrak{N}_-)$$

defines an anti-isomorphism of $Q_L(\mathfrak{N}_-)$ and $Q_R(\mathfrak{N}_-)$, which is denoted by θ too.

LEMMA 3. $\theta: Q_L(\mathfrak{N}_-) \to Q_R(\mathfrak{N}_-)$ maps $u_L(w_0 w w_0, w_0\lambda)$ into $u_R(w, \lambda)^{-1}$, for all $w \in W$ and $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$.

PROOF. By induction on $\mathfrak{l}(w)$.

If $\mathfrak{l}(w) = 1$, then $w = s_\alpha$, for some $\alpha \in S$. Let $k = \lambda(H_\alpha)$, then:

$$\theta(u_L(w_0 s_\alpha w_0, w_0\lambda)) = \theta(\mathbb{C}X^{-k}_{w_0(\alpha)}) = \mathbb{C}X^{-k}_{-\alpha} = u_R(s_\alpha, \lambda)^{-1}.$$

If $\mathfrak{l}(w) > 1$, then there exist $\alpha \in S$ and $w' \in W$, such that $\mathfrak{l}(w') < \mathfrak{l}(w)$ and $w' s_\alpha = w$. Suppose the lemma holds for w' and all $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$; then

$$\begin{aligned}\theta(u_L(w_0 w w_0, w_0\lambda)) &= \theta(u_L(w_0 w' w_0, w_0 s_\alpha(\lambda)) \cdot u_L(w_0 s_\alpha w_0, w_0\lambda)) = \\ &= \theta(u_L(w_0 s_\alpha w_0, w_0 s_\alpha(\lambda))) \cdot \theta(u_L(w_0 w' w_0, w_0\lambda)) = u_R(s_\alpha, \lambda)^{-1} \cdot u_R(w', \lambda)^{-1} = \\ &= u_R(w, \lambda)^{-1}.\end{aligned}$$

COROLLARY 1. Let $w \in W$ and $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$. Then $M_{w(\lambda)} \subset M_\lambda$ if and only if $M_{w_0(\lambda)} \subset M_{w_0 w(\lambda)}$.

PROOF. Clear from lemma 3 and theorem 2; note that $u(w^{-1}, w\lambda) = u(w, \lambda)^{-1}$, for all $w \in W$ and $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$.

The next lemma is a slight extension of lemma 9 of [2]. (The proof remains the same.)

LEMMA 4. Suppose $M_\mu \subset M_\lambda$ and $\alpha \in S$ such that $\mu(H_\alpha)$ is a nonpositive integer, then $M_\mu \subset M_{s_\alpha(\lambda)}$.

PROOF. By lemma 1, there are two possibilities, $M_{s_\alpha(\lambda)} \subset M_\lambda$ or $M_{s_\alpha(\lambda)} \supset M_\lambda$. The latter case is trivial, so we suppose that $M_{s_\alpha(\lambda)} \subset M_\lambda$. Suppose $u \in U(\mathfrak{R}_-)$ is such that uf_λ is $\mathfrak{R}_+$-extreme and has weight $\lambda - \delta$. It is possible to write $X^k_{-\alpha} \cdot u$ as $u_1 \cdot X^{k_1}_{-\alpha}$, for some $u_1 \in U(\mathfrak{R}_-)$ and a nonnegative integer k_1, which increases unboundedly with k. Hence $X^k_{-\alpha} \cdot uf_\lambda \in M_{s_\alpha(\lambda)}$, for some k. Suppose k minimal with this property. By lemma 1, we have $X_\alpha \cdot X^k_{-\alpha} uf_\lambda = -k(\mu({}_\alpha H) - k) \cdot X^{k-1}_{-\alpha} uf_\lambda$; hence $k = 0$, so $uf_\lambda \in M_{s_\alpha(\lambda)}$.

COROLLARY 2. If $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$ is such that $\lambda(H_\alpha)$ is a nonnegative integer for some $\alpha \in S$, then $M_{s_\alpha(\mu)} \subset M_\lambda$ if $M_\mu \subset M_\lambda$.

PROOF. If we set $\mu' = w_0(\lambda)$, $\lambda' = w_0(\mu)$ and $\alpha' = -w_0(\alpha)$, then $M_{\mu'} \subset M_{\lambda'}$ by corollary 1. Because $\mu'(H_{\alpha'})$ is a nonpositive integer, it follows then from lemma 4 that $M_{\mu'} \subset M_{s_{\alpha'}(\lambda')}$ and hence from corollary 1 again: $M_{s_\alpha(\mu)} = M_{w_0 s_{\alpha'}(\lambda')} \subset M_{w_0(\mu')} = M_\lambda$.

THEOREM 3. Suppose $\lambda \in \mathfrak{h}^*_{\mathbf{Z}} \cap \bar{C}$ and $d, d' \in D_{S(\lambda)}$. If $M_{d(\lambda)} \subset M_{d'(\lambda)}$, then $d \geqslant d'$.

PROOF. If $\mathfrak{l}(d) = 0$, then $d = 1$; hence it follows from the fact that $d'(\lambda) - \lambda$ is a sum of positive roots and [1, Chap. VI, no. 1.6, prop. 18] that $d'(\lambda) = \lambda$, and hence that $d' = 1$; see [1, exercise 3 on p. 37]. Now, let $d \neq 1$ and suppose that $M_{d''(\lambda)} \subset M_{d^\wedge(\lambda)}$ implies $d'' \geqslant d^\wedge$ if $d'', d^\wedge \in D_{S(\lambda)}$ and $\mathfrak{l}(d'') < \mathfrak{l}(d)$. Because $\mathfrak{l}(d) \neq 0$, there is an $\alpha \in S$ such that $\mathfrak{l}(s_\alpha d) < \mathfrak{l}(d)$, so $d'' = s_\alpha d < d$ and $d'' \in D_{S(\lambda)}$. By lemma 4 we then have: $M_{d(\lambda)} \subset M_{d'(\lambda)}$ and $M_{d(\lambda)} \subset M_{s_\alpha d'(\lambda)}$. If $d'(\lambda)(H_\alpha) \geqslant 0$, it follows by corollary 2 that $M_{d''(\lambda)} \subset M_{d'(\lambda)}$ and by induction: $d'' \geqslant d'$. Hence $d \geqslant d'$.

If $d'(\lambda)(H_\alpha) < 0$, then by lemma 2: $s_\alpha d' < d'$ and $s_\alpha d' \in D_{S(\lambda)}$. By corollary 2 and the fact that $s_\alpha d'(\lambda)(H_\alpha) = -d'(\lambda)(H_\alpha) > 0$, it follows that $M_{d''(\lambda)} \subset M_{s_\alpha d'(\lambda)}$; hence it follows by induction that $d'' \geqslant s_\alpha d'$ and by (**) that $d \geqslant d'$.

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